

Characteristics polynomial of normalized Laplacian for trees

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Abstract

Here, we find the characteristics polynomial of normalized Laplacian of a tree. The coefficients of this polynomial are expressed by the higher order general Randić indices for matching, whose values depend on the structure of the tree. We also find the expression of these indices for starlike tree and a double-starlike tree, $H_m(p, q)$. Moreover, we show that two cospectral $H_m(p, q)$ of the same diameter are isomorphic.

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1 Introduction

Let $\Gamma = (V, E)$ be a simple finite undirected graph of order n . Two vertices $u, v \in V$ are called neighbours, $u \sim v$, if they are connected by an edge in E , $u \not\sim v$ otherwise. Let d_v be the degree of a vertex $v \in V$, that is, the number of neighbours of v . The *normalized Laplacian matrix* [7], \mathcal{L} , of Γ is defined as:

$$\mathcal{L}(\Gamma)_{u,v} = \begin{cases} 1 & \text{if } u = v \text{ and } d_v \neq 0, \\ -\frac{1}{\sqrt{d_u d_v}} & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

This \mathcal{L} is similar to the normalized Laplacian Δ defined in [2, 20]. Let $\phi_\Gamma(x) = \det(xI - \mathcal{L})$ be the characteristics polynomial of $\mathcal{L}(\Gamma)$. Let us consider $\phi_\Gamma(x) = a_0 x^n - a_1 x^{n-1} + a_2 x^{n-2} - \dots + (-1)^{n-1} a_{n-1} x + (-1)^n a_n$. Now if Γ has no isolated vertices then $a_0 = 1$, $a_1 = n$,

$a_2 = \frac{n(n-1)}{2} - \sum_{i \sim j} \frac{1}{d_i d_j}$ and $a_n = 0$ (for some properties of $\phi_\Gamma(x)$ see [8]). The zeros of $\phi_\Gamma(x)$ are the eigenvalues of \mathcal{L} and we order them as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = 0$. Γ is connected iff $\lambda_{n-1} > 0$. $\lambda_1 \leq 2$, the equality holds iff Γ has a bipartite component. Moreover, Γ is bipartite iff for each λ_i , the value $2 - \lambda_i$ is also an eigenvalue of Γ . See [7] for more properties of the normalized Laplacian eigenvalues.

1.1 General Randić index for matching

There are different topological indices. The degree based topological indices [11, 21], which include *Randić index* [23], reciprocal *Randić index* [16], general *Randić index* [4], higher order *connectivity index* [3, 22, 24], *connective eccentricity index* [29, 30], *Zagreb index* [1, 9, 12, 13, 14, 18, 27, 28] are more popular than others.

For any real number α , the general Randić index of a graph Γ is defined by B. Bollobás and P. Erdős (see [4]) as:

$$R_\alpha(\Gamma) = \sum_{i \sim j} (d_i d_j)^\alpha, \quad (2)$$

which is the general expression of the Randić index (also known as connectivity index) introduced by M. Randić in 1975 [23] by choosing $\alpha = -1/2$ in (2). For more properties of the general Randić index of graphs, we refer to [5, 16, 17, 24, 25, 26].

The Zagreb index of a graph was first introduced by Gutman et al. [12] in 1972. For a graph Γ the first and the second Zagreb indices are defined by

$$Z_1(\Gamma) = \sum_{i \in V} d_i^2 \text{ and } Z_2(\Gamma) = \sum_{i \sim j} d_i d_j,$$

respectively. Now, for any positive integer p , we define the p^{th} order general *Randić index for matching* as

$$R_\alpha^{(p)}(\Gamma) = \sum_{M_p \in \mathcal{M}_p(\Gamma)} \prod_{e \in M_p} s(e)^\alpha, \quad (3)$$

where $s(e) = d_u d_v$ is the *strength* of the edge $e = uv \in E$, M_p is a p -matching, that is, a set of p non-adjacent edges and $\mathcal{M}_p(\Gamma)$ is the set of all p -matchings in Γ . The first order general Randić index for matching with $\alpha = 1$ is the second Zagreb index, that is, $Z_2(\Gamma) = R_1^{(1)}(\Gamma)$. We take $R_\alpha(\Gamma) = R_\alpha^{(1)}(\Gamma)$ and

$$R_\alpha^{(0)}(\Gamma) = \begin{cases} 0 & \text{if } \Gamma \text{ is the null graph,} \\ 1 & \text{otherwise.} \end{cases}$$

If Γ is r -regular, then $R_\alpha^{(i)} = r^{2i\alpha} |\mathcal{M}_i(\Gamma)|$. The $R_{-1}^{(2)}$ for some known graphs are as follows: $R_{-1}^{(2)}(S_n) = 0$, $R_{-1}^{(2)}(P_n) = \frac{n^2 - n - 4}{32}$, $R_{-1}^{(2)}(C_n) = \frac{n(n-3)}{32}$, $R_{-1}^{(2)}(K_{p,q}) = \frac{(p-1)(q-1)}{4pq}$, and $R_{-1}^{(2)}(K_n) = \frac{3\binom{n}{4}}{(n-1)^4}$, where the notations, S_n , P_n , C_n , K_n and $K_{p,q}$ have their usual meanings.

Theorem 1.1. *For any real number α ,*

$$0 \leq R_\alpha^{(2)}(\Gamma) \leq \frac{1}{2} \left[R_\alpha(\Gamma) \right]^2 - \frac{1}{2} R_{2\alpha}(\Gamma)$$

Proof.

$$\begin{aligned} \left[R_\alpha(\Gamma) \right]^2 &= \left[\sum_{e \in E} (s(e))^\alpha \right]^2 \\ &= \sum_{e \in E} (s(e))^{2\alpha} + 2 \sum_{e_1, e_2 \in \mathcal{M}_2(\Gamma)} (s(e_1)s(e_2))^\alpha + 2 \sum_{e_1, e_2 \notin \mathcal{M}_2(\Gamma)} (s(e_1)s(e_2))^\alpha \end{aligned}$$

which proves our required result. \square

Clearly, for any two graphs Γ_1, Γ_2 , and $p \geq 0$, $R_\alpha^{(p)}(\Gamma_1 \cup \Gamma_2) \geq R_\alpha^{(p)}(\Gamma_1) + R_\alpha^{(p)}(\Gamma_2)$. The equality holds, when $p = 1$ or one of the graphs is null.

It has been seen that the matching plays a role in the spectrum of a tree. In [6], some results on normalized Laplacian spectrum for trees have been discussed. Now, we derive (or express the coefficients of) the characteristics polynomial $\phi_T(x)$ of a tree T in terms of $R_{-1}^{(i)}(T)$.

2 The characteristics polynomial of normalized Laplacian for a tree

Theorem 2.1. *Let T be a tree with n vertices and maximum matching number k , then*

$$\phi_T(x) = \sum_{i=0}^k (-1)^i (x-1)^{n-2i} R_{-1}^{(i)}(T) \quad (4)$$

and the coefficients of ϕ_T are given by

$$a_p = \sum_{i=0}^k (-1)^i \binom{n-2i}{p-2i} R_{-1}^{(i)}(T). \quad (5)$$

Proof. Consider a matrix, $B = [b_{ij}] = xI_n - \mathcal{L}$, where

$$b_{ij} = \begin{cases} x-1 & \text{if } i=j \\ \frac{1}{\sqrt{d_i d_j}} & \text{if } i \sim j \\ 0 & \text{else.} \end{cases}$$

Now,

$$\begin{aligned} \phi_T(x) &= \det(B) \\ &= \sum_{\sigma \in \mathcal{S}_n} b_\sigma, \end{aligned}$$

where $b_\sigma = \text{sgn}(\sigma)b_{1,\sigma(1)}b_{2,\sigma(2)}\cdots b_{n,\sigma(n)}$ and \mathcal{S}_n is the set of all permutation of $\{1, \dots, n\}$. Now, for any $\sigma \in \mathcal{S}_n$ and $\sigma(i) \neq i$, $b_\sigma \neq 0$ only when $i \sim \sigma(i)$. Since, T does not contain any cycle, here, σ is either the identity permutation or a product of disjoint transpositions. When σ is the identity permutation, $b_\sigma = (x-1)^n$ and if $\sigma = (i_1\sigma(i_1))(i_2\sigma(i_2))\cdots(i_l\sigma(i_l))$ is a product of disjoint transpositions, then,

$$b_\sigma = \begin{cases} (-1)^l(x-1)^{n-2l} \frac{1}{d_{i_1}d_{\sigma(i_1)}} \frac{1}{d_{i_2}d_{\sigma(i_2)}} \cdots \frac{1}{d_{i_l}d_{\sigma(i_l)}} & \text{if } i_j \sim \sigma(i_j) \ \forall j \\ 0 & \text{otherwise.} \end{cases}$$

Now, since, T has the maximum matching number k ,

$$\begin{aligned} \phi_T(x) &= \sum_{i=0}^k \sum_{M \in \mathcal{M}_i} (-1)^{|M|} (x-1)^{n-2|M|} \prod_{e \in M} \frac{1}{s(e)} \\ &= \sum_{i=0}^k (-1)^i (x-1)^{n-2i} R_{-1}^{(i)}(T). \end{aligned}$$

Expanding the right hand side of the above equation we get

$$a_p = \sum_{i=0}^k (-1)^i \binom{n-2i}{p-2i} R_{-1}^{(i)}(T).$$

□

Corollary 2.1. *For a tree T with maximum matching number k ,*

$$1 - R_{-1}(T) + R_{-1}^{(2)}(T) - \cdots + (-1)^k R_{-1}^{(k)}(T) = 0. \quad (6)$$

Corollary 2.2. *Let T be a tree with maximum matching number k . The eigenvalues of T are 1 with the multiplicity $n-2k$, and $1 \pm \sqrt{\alpha_i}$ ($1 \leq i \leq k$) where α_i 's are the zeros of the polynomial*

$$\psi_T(y) = y^k - R_{-1}(T)y^{k-1} + \cdots + (-1)^k R_{-1}^{(k)}(T). \quad (7)$$

The characteristics polynomial $\phi_T(x)$ of a tree T can be expressed in terms of $R_{-1}^{(i)}(T)$, whose value depends on the structure of T . Now, we find the expression of $R_{-1}^{(i)}(T)$ for two different trees, starlike tree [10, 22] and a specific type of double starlike trees [15, 19].

2.1 Starlike tree

A tree is called *starlike* tree (see Figure 2.1) if it has exactly one vertex v of degree grater than two. We denote a starlike tree with $d_v = r$, $3 \leq r \leq n-1$, by $T(l_1, l_2, \dots, l_r)$ where l_i 's are positive integers with $l_1 + l_2 + \cdots + l_r = n-1$, that is, $T(l_1, l_2, \dots, l_r) - v = P_{l_1} \cup P_{l_2} \cup \cdots \cup P_{l_r}$ where P_{l_i} is, a path on l_i vertices, connected to v (see figure (1) for an example). Now onwards, without loss of any generality, we assume $1 \leq l_1 \leq l_2 \leq \cdots \leq l_r$.

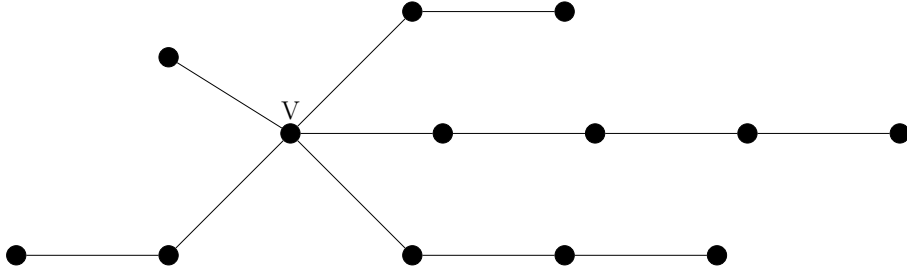


Figure 1: Starlike tree, $T(1, 2, 2, 3, 4)$

Theorem 2.2. Let $T(l_1, l_2, \dots, l_r)$ be a starlike tree on n vertices with maximum matching number k . If there are exactly m ($m \neq 0$) number of l_i 's which are odd numbers then

(i) T has the maximum matching number $\frac{n-m+1}{2}$,

(ii) $R_{-1}(T) = \frac{1}{4}(n+1) + \frac{m}{4}(\frac{2}{r} - 1)$.

Furthermore, if $l_{p_1}, l_{p_2}, \dots, l_{p_m}$ are odd then

(iii) $|\mathcal{M}_k(T)| = \frac{\prod (l_{p_j} + 1)}{2^{m-1}} \cdot \sum_{j=1}^m \frac{1}{l_{p_j} + 1}$, and

(iv) $R_{-1}^{(k)}(T) = \frac{\prod l_{p_j}}{\prod d_i} \cdot \sum_{j=1}^m \frac{1}{l_{p_j}}$.

Proof. (i) This part is obvious, since a maximal matching can be taken as follows: for each even l_i , we cover the corresponding P_{l_i} by a perfect matching. Consider another perfect matching on a $P_{l_{i+1}}$ where $v \in P_{l_{i+1}}$ and l_i is odd. For all other odd l_i 's, take $\frac{l_i-1}{2}$ matching.

(ii) T has m edges of strength r , $(r-m)$ edges of strength $2r$, $(r-m)$ edges of strength 2 and the rest $(n-2r+m-1)$ edges are of strength 4. Thus,

$$\begin{aligned} R_{-1}(T) &= \sum_{e \in E} \frac{1}{s(e)} \\ &= \frac{1}{4}(n+1) + \frac{m}{4}(\frac{2}{r} - 1). \end{aligned}$$

(iii) P_x has maximum matching number $\frac{x-1}{2}$ with $\frac{x+1}{2}$ number of matchings, when x is odd. Thus,

$$\begin{aligned} |\mathcal{M}_k(T)| &= \sum_{j=1}^m \prod_{\substack{k=1 \\ k \neq j}}^m \frac{(l_{p_k} + 1)}{2} \\ &= \frac{\prod (l_{p_j} + 1)}{2^{m-1}} \cdot \sum_{j=1}^m \frac{1}{l_{p_j} + 1}. \end{aligned}$$

- (iv) To get a maximal matching in $T(l_1, l_2, \dots, l_r)$, where $l_{p_1}, l_{p_2}, \dots, l_{p_m}$ are odd, one l_{p_j} is combined with v_1 and the in other odd l_{p_j} one vertex, of degree one or two, remains uncovered. One or $\frac{l_{p_j}-1}{2}$ positions are possible if the degree of the uncover vertex is one or two respectively. Thus,

$$\begin{aligned}
R_{-1}^{(k)}(T) &= \frac{1}{\prod d_i} \left[m + (m-1) \cdot 2 \sum_{j=1}^m \frac{(l_{p_j} - 1)}{2} + \dots + 2^{k-1} \sum_{j=1}^m \prod_{\substack{k=1 \\ k \neq j}}^m \frac{(l_{p_k} - 1)}{2} \right] \\
&= \frac{1}{\prod d_i} \left[\sum_{j=1}^m \prod_{\substack{k=1 \\ k \neq j}}^m (1 + (l_{p_k} - 1)) \right] \\
&= \frac{\prod l_{p_j}}{\prod d_i} \cdot \sum_{j=1}^m \frac{1}{l_{p_j}}.
\end{aligned}$$

□

Corollary 2.3. *If T is a tree as in theorem (2.2), then the multiplicity of the eigenvalue 1 is $m - 1$.*

Remark. If $m = 0$ in theorem (2.2), then T has maximum matching number $k = \frac{n-1}{2}$ with $|\mathcal{M}_k(T)| = \frac{n+1}{2}$ and $R_{-1}^{(k)}(T) = \frac{n-1}{\prod d_i}$.

Example: The spectrum of different starlike trees with $n = 8$ vertices are given bellow. Superscripts in the table show the algebraic multiplicity of an eigenvalue.

	Partition	Randić Indices	$\lambda(\mathcal{L})$
1.	1,1,5	$\frac{25}{12}, \frac{21}{16}, \frac{11}{48}$	$0, 2, 1^2, 1 \pm \frac{\sqrt{13 \pm \sqrt{37}}}{2\sqrt{6}}$
2.	1,2,4	$\frac{13}{6}, \frac{3}{2}, \frac{17}{48}, \frac{1}{48}$	$0, 2, 1 \pm 0.876, 1 \pm 0.558, 1 \pm 0.295$
3.	1,3,3	$\frac{13}{6}, \frac{71}{48}, \frac{5}{16}$	$0, 2, 1^2, 1 \pm \frac{\sqrt{3}}{2}, 1 \pm \frac{\sqrt{5}}{2\sqrt{3}}$
4.	2,2,3	$\frac{9}{4}, \frac{5}{3}, \frac{7}{16}, \frac{1}{48}$	$0, 2, 1 \pm \frac{1}{\sqrt{2}}, 1 \pm \frac{\sqrt{9 \pm \sqrt{57}}}{2\sqrt{6}}$
5.	1,1,1,4	$\frac{15}{8}, \frac{31}{32}, \frac{3}{32}$	$0, 2, 1^2, 1 \pm \frac{\sqrt{3}}{2}, 1 \pm \frac{1}{2\sqrt{2}}$
6.	1,1,2,3	$2, \frac{39}{32}, \frac{7}{32}$	$0, 2, 1^2, 1 \pm \frac{\sqrt{4 \pm \sqrt{2}}}{2\sqrt{2}}$
7.	1,2,2,2	$\frac{17}{8}, \frac{3}{2}, \frac{13}{32}, \frac{1}{32}$	$0, 2, (1 \pm \frac{1}{\sqrt{2}})^2, 1 \pm \frac{1}{2\sqrt{2}}$
8.	1,1,1,1,3	$\frac{33}{20}, \frac{13}{20}$	$0, 2, 1^4, 1 \pm \sqrt{\frac{13}{20}}$
9.	1,1,1,2,2	$\frac{9}{5}, \frac{19}{20}, \frac{3}{20}$	$0, 2, 1^2, 1 \pm \sqrt{\frac{3}{10}}, 1 \pm \frac{1}{\sqrt{2}}$
10.	1,1,1,1,1,2	$\frac{17}{12}, \frac{5}{12}$	$0, 2, 1^4, 1 \pm \sqrt{\frac{5}{12}}$

2.2 Double starlike tree

A tree T is called *double starlike* if it has exactly two vertices of degree greater than two. Let $H_m(p, q)$ be a double starlike tree obtained by attaching p pendant vertices to one end-vertex of a path P_m and q pendant vertices to the other end-vertex of P_m . Thus, $H_2(p, q)$ is

a double star $S_{p+1,q+1}$, that is, a tree with exactly two non-pendant vertices with the degree $p + 1$ and $q + 1$ respectively. See figure (2) for examples.

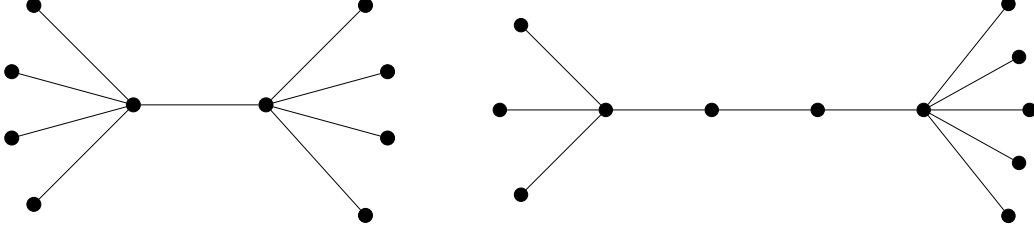


Figure 2: Double starlike tree $H_2(4,4)$ and $H_4(3,5)$

Lemma 2.1. *Let $C_{i,n}$ and k be the number of i -matchings and maximal matching number, respectively, of P_n ($n \geq 3$), then*

$$R_{-1}^{(i)}(P_n) = \frac{1}{4^i} C_{i,n-2} + \frac{1}{4^{i-1}} C_{i-1,n-2}, \text{ and}$$

$$\psi_{P_n}(y) = (y-1) \sum_{i=0}^{k-1} (-1)^i \frac{1}{4^i} C_{i,n-2} y^{k-1-i},$$

where $\psi_{P_n}(y)$ is defined as in corollary (2.2).

Proof. The maximum matching number of P_n is $\lfloor \frac{n}{2} \rfloor$ and $C_{i,n} = \binom{n-i}{i}$. P_n has $n-3$ edges of strength 4 and two edges of strength 2. Thus,

$$\begin{aligned} R_{-1}^{(i)}(P_n) &= \frac{1}{4^i} C_{i,n-2} + 2 \times \frac{1}{2} \times \frac{1}{4^{i-1}} C_{i-1,n-3} + \frac{1}{4^{i-1}} C_{i-2,n-4} \\ &= \frac{1}{4^i} C_{i,n-2} + \frac{1}{4^{i-1}} C_{i-1,n-2}. \end{aligned}$$

which proves the first part of the theorem.

For the second part of the lemma, we have, $R_{-1}^{(k)}(P_n) = \frac{1}{4^{k-1}} C_{k-1,n-2}$.

Therefore,

$$\begin{aligned} \psi_{P_n}(y) &= (y^k - y^{k-1}) - \frac{1}{4} C_{1,n-2} (y^{k-1} - y^{k-2}) + \dots \\ &\quad + (-1)^{k-1} \frac{1}{4^{k-1}} C_{k-1,n-2} (y-1) \\ &= (y-1) \sum_{i=0}^{k-1} (-1)^i \frac{1}{4^i} C_{i,n-2} y^{k-1-i}. \end{aligned}$$

□

Theorem 2.3. *Let T be the double starlike tree $H_m(p,q)$, then*

- (i) T has the maximum matching number $\lfloor \frac{m}{2} \rfloor + 1$, and the multiplicity of the eigenvalue 1 is $p+q-2$ if m is even and $p+q-1$ if m is odd,

$$(ii) \quad R_{-1}^{(i)}(T) = R_{-1}^{(i)}(P_m) + \frac{pq}{(p+1)(q+1)} R_{-1}^{(i-1)}(P_m) + \frac{p+q}{2(p+1)(q+1)} R_{-1}^{(i-1)}(P_{m-1}),$$

(iii) and

$$\psi_T(y) = \begin{cases} (y - r_1)\psi_{P_m}(y) - r_2\psi_{P_{m-1}}(y) & \text{if } m \text{ odd,} \\ y(\psi_{P_m}(y) - r_2\psi_{P_{m-1}}(y)) - r_1\psi_{P_m}(y) & \text{if } m \text{ even,} \end{cases} \quad (8)$$

where $r_1 = \frac{pq}{(p+1)(q+1)}$, $r_2 = \frac{p+q}{2(p+1)(q+1)}$ and $\psi_T(y)$ is defined as in corollary (2.2).

Proof. (i) This is easy to verify.

(ii) When $m \geq 3$, T has p edges of strength $p+1$, q edges of strength $q+1$, 1 edge of strength $2(p+1)$, 1 edges of strength $2(q+1)$ and rest $m-3$ edges of strength 4. Thus,

$$\begin{aligned} R_{-1}^{(i)}(T) &= \frac{1}{4^i} C_{i,m-2} + \frac{1}{4^{i-1}} \left[\frac{p}{p+1} + \frac{q}{q+1} \right] C_{i-1,m-2} + \frac{1}{4^{i-1}} \left[\frac{1}{2(p+1)} + \frac{1}{2(q+1)} \right] C_{i-1,m-3} \\ &\quad + \frac{1}{4^{i-2}} \frac{pq}{(p+1)(q+1)} C_{i-2,m-2} + \frac{1}{4^{i-2}} \frac{p+q}{2(p+1)(q+1)} C_{i-2,m-3} \\ &\quad + \frac{1}{4^{i-2}} \frac{1}{4(p+1)(q+1)} C_{i-2,m-4} \\ &= \frac{1}{4^i} C_{i,m-2} + \frac{1}{4^{i-1}} C_{i-1,m-2} + \frac{1}{4^{i-1}} \frac{pq}{(p+1)(q+1)} C_{i-1,m-2} \\ &\quad + \frac{1}{4^{i-2}} \frac{pq}{(p+1)(q+1)} C_{i-2,m-2} + \frac{1}{4^{i-1}} \frac{p+q}{2(p+1)(q+1)} C_{i-1,m-3} \\ &\quad + \frac{1}{4^{i-2}} \frac{p+q}{2(p+1)(q+1)} C_{i-2,m-3} \\ &= R_{-1}^{(i)}(P_m) + \frac{pq}{(p+1)(q+1)} R_{-1}^{(i-1)}(P_m) + \frac{p+q}{2(p+1)(q+1)} R_{-1}^{(i-1)}(P_{m-1}) \end{aligned}$$

Again if $m = 2$, then

$$\begin{aligned} R_{-1}(T) &= 1 + \frac{pq}{(p+1)(q+1)}, \text{ and} \\ R_{-1}^{(2)}(T) &= \frac{pq}{(p+1)(q+1)}. \end{aligned}$$

(iii) We have, $\psi_T(y) = y^k - R_{-1}(T)y^{k-1} + \dots + (-1)^k R_{-1}^{(k)}(T)$ and $R_{-1}^{(i)}(T) = R_{-1}^{(i)}(P_m) + r_1 R_{-1}^{(i-1)}(P_m) + r_2 R_{-1}^{(i-1)}(P_{m-1})$, $1 \leq i \leq k$, where k is the maximum matching number of T . Hence the maximum matching number of P_m is $k-1$ and the same is of P_{m-1} is $k-1$ if m odd and $k-2$ otherwise.

Thus,

$$R_{-1}^{(k)}(T) = \begin{cases} r_1 R_{-1}^{(k-1)}(P_m) + r_2 R_{-1}^{(k-1)}(P_{m-1}) & \text{if } m \text{ odd,} \\ r_1 R_{-1}^{(k-1)}(P_m) & \text{if } m \text{ even.} \end{cases} \quad (9)$$

Hence the result follows. □

Example: Consider the double starlike trees as in figure (2). For the first tree, $T_1 = H_2(4, 4)$ we have $R_{-1}^{(1)}(T_1) = \frac{41}{25}$ and $R_{-1}^{(2)}(T_1) = \frac{16}{25}$. Thus the eigenvalues of T_1 are 0, 1^6 , 2, 0.2, 1.8. The second tree, $T_2 = H_4(3, 5)$ has maximum matching number equals to 3 and the Randić indices of matching are $R_{-1}^{(1)}(T_2) = \frac{49}{24}$, $R_{-1}^{(2)}(T_2) = \frac{115}{96}$ and $R_{-1}^{(3)}(T_2) = \frac{5}{32}$. Hence the eigenvalues of T_2 are 0, 1^6 , 2, 1 ± 0.4263 , 1 ± 0.9273 .

Theorem 2.4. Let $T_1 = H_m(p_1, q_1)$ and $T_2 = H_m(p_2, q_2)$ be \mathcal{L} -cospectral. Then, T_1 and T_2 are isomorphic.

Proof. From theorem (2.3) and (2.1) we have, $p_1 + q_1 = p_2 + q_2$ and $p_1q_1 = p_2q_2$. Hence the proof. □

3 Summary and Conclusions

The Zegreb indeices and the Randić index are of great importance for molecular chemistry. They are used to characterize the molecular branching in chemical graphs. The general Randić indices for matchng can also play an important role in this area. It can be used to characterize different classes of graphs. The estimation of general Randić indices for matching for trees, which are the simplest structure amongst the graphs, is much easier than others. Corollary (2.1) states that, for a n -vertex tree with maximum matching number k it is suffitient to calculate the zeros of a k degree polynomial to determine its complete spectrum. Furthermore, the corollay (2.1) shows that the general Randić indices are related by a simple equation. Thus, we only need to calculate the $k - 1$ general Randić indices for matching to compute the complete set of eigenvalues.

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References

- [1] H. Abdo, D. Dimitrov, T. Reti, D. Stevanovic, Estimating the Spectral Radius of a Graph by the Second Zagreb Index, MATCH Commun. Math. Comput. Chem. 72 (2014) 741-751.
- [2] A. Banerjee, J. Jost. On the spectrum of the normalized graph Laplacian, Linear Algebra Appl. 428 (2008) 3015-3022.
- [3] Y. Alizadeh, On the Higher Randić Index, Iranian J. Mathematical Chem. 4(2) (2013) 257-263.

- [4] B. Bollobás and P. Erdős, Graphs of extremal weights, *Ars Combin.* 50 (1998) 225-233.
- [5] M. Cavers, S. Fallat, S. Kirkland, On the normalized Laplacian energy and general Randić index R_{-1} of graphs, *Linear Algebra Appl.* 433 (2010) 172-190.
- [6] H. Chen, J. Jost, Minimum vertex covers and the spectrum of the normalized Laplacian on trees, *Linear Algebra Appl.* 437 (2012) 1089-1101 .
- [7] F.Chung, *Spectral graph theory*, AMS, 1997.
- [8] J-M. Guo, J. Li, W.C. Shiu, On the Laplacian, signless Laplacian and normalized Laplacian characteristic polynomials of a graph, *Czechoslovak Mathematical Journal* 63 (3) (2013) 701-720.
- [9] I. Gutman, An exceptional property of first Zagreb index, *MATCH Commun. Math. Comput. Chem.* 72 (2014) 733-740.
- [10] I. Gutman, O. Araujo, J. Rada, Matchings in starlike trees, *Appl. Math. Lett.* 14 (2001) 843-848.
- [11] I. Gutman, B. Furtula, C. Elphick, Three new/old vertex-degree-based topological indices, *MATCH Commun. Math. Comput. Chem.* 72 (2014) 617-632.
- [12] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total π -electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* 17 (1972) 535-538.
- [13] A. Hamzeh, T. Réti, An Analogue of Zagreb Index Inequality Obtained from Graph Irregularity Measures, *MATCH Commun. Math. Comput. Chem.* 72 (2014) 669-683.
- [14] R. Kazemi, The Second Zagreb Index of Molecular Graphs with Tree Structure, *MATCH Commun. Math. Comput. Chem.* 72 (2014) 753-760.
- [15] M. Lazić, Some results on symmetric double starlike trees, *J. Appl. Math. Comput.* 21 (2006) 215-222.
- [16] X. Li, Y. Shi, A survey on the Randić index, *MATCH Commun. Math. Comput. Chem.* 59 (2008) 127-156.
- [17] X. Li, Y. Shi, On a relation between the Randić index and the chromatic number, *Discrete Math.* 310 (2010) 2448-2451.
- [18] H. Lin, Vertices of degree two and the first Zagreb index of trees, *MATCH Commun. Math. Comput. Chem.* 72 (2014) 825-834.
- [19] X.G. Liu, Y.P. Zhang, P.L. Lu, One special double starlike graphs is determined by its Laplacian spectra, *Appl. Math. Lett.* 22 (2009) 435-438.
- [20] R. Mehatari, A. Banerjee, Effect on normalized graph Laplacian spectrum by motif attachment and duplication, *Appl. Math. Comput.* 261 (2015) 382-387.

- [21] J. Rada, R. Cruz, Vertex-degree-based topological indices over graphs, MATCH Commun. Math. Comput. Chem. 72 (2014) 603-616.
- [22] J. Rada and O. Araujo, Higher order connectivity index of star-like trees, Discrete Appl. Math. 119 (2002) 287-295.
- [23] M. Randić, On characterization of molecular branching, J. Amer. Chem. Soc. 97 (1975) 6609-6615.
- [24] J. A. Rodriguez, A spectral approach to the Randi index, Linear Algebra Appl. 400 (2005) 399-344.
- [25] L. Shi, Bounds on Randić indices, Discrete Math. 309 (2009) 5238-5241.
- [26] Y. Shi, Note on two generalizations of the Randić index, Appl. Math. Comput. 265 (2015) 1019-1025.
- [27] A. Vasilyev, R. Darda, D. Stevanovic, Trees of given order and independence number with minimal first Zagreb index, MATCH Commun. Math. Comput. Chem. 72 (2014) 775-782.
- [28] K. Xu, K.C. Das, S. Balachandran, Maximizing the Zagreb indices of (n, m) -graphs, MATCH Commun. Math. Comput. Chem. 72 (2014) 641-654.
- [29] G. Yu, L. Feng, On the connective eccentricity index of graphs, MATCH Commun. Math. Comput. Chem. 69 (2013) 611-628.
- [30] G. Yu, H. Qu, L. Tang, L. Feng, On the connective eccentricity index of trees and unicyclic graphs with given diameter, J. Math. Anal. Appl. 420 (2014) 1776- 1786.